

Hydrodynamic damping in trapped Bose gases

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Abstract

Griffin, Wu and Stringari have derived the hydrodynamic equations of a trapped dilute Bose gas above the Bose-Einstein transition temperature. We give the extension which includes hydrodynamic damping, following the classic work of Uehling and Uhlenbeck based on the Chapman-Enskog procedure. Our final result is a closed equation for the velocity fluctuations $\delta\mathbf{v}$ which includes the hydrodynamic damping due to the shear viscosity η and the thermal conductivity κ . Following Kavoulakis, Pethick and Smith, we introduce a spatial cutoff in our linearized equations when the density is so low that the hydrodynamic description breaks down. Explicit expressions are given for η and κ , which are position-dependent through dependence on the local fugacity when one includes the effect of quantum degeneracy of the trapped gas. We also discuss a trapped Bose-condensed gas, generalizing the work of Zaremba, Griffin and Nikuni to include hydrodynamic damping due to the (non-condensate) normal fluid.

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I. INTRODUCTION

Recent observation of Bose-Einstein condensation in trapped atomic gases has stimulated interest in the collective oscillations of non-uniform Bose gases [1,2]. The hydrodynamic equations of a trapped Bose gas (neglecting damping) have been recently derived above [3] and below [4] the Bose-Einstein transition temperature (T_{BEC}). These hydrodynamic equations describe low frequency phenomena ($\omega \ll 1/\tau_c$, where τ_c is the mean time between collisions of atoms), when collisions are sufficiently strong to ensure local thermodynamic equilibrium. These results can be used to derive closed equations for the velocity fluctuations in a trapped gas, whose solution gives the normal modes of oscillation. In particular, above T_{BEC} , Ref. [3] gives explicit results for the frequencies of surface, monopole, and coupled monopole-quadrupole modes. Available experimental data on the frequencies and damping of collective oscillations in trapped gases [1,2] is mainly for the collisionless regime, rather than the hydrodynamic regime we consider in this paper. However sufficiently high densities which allow one to probe the hydrodynamic region at finite temperatures are expected in the near future (see also Refs. [5,6]). Thus it is worthwhile to extend the discussion of hydrodynamic modes in Refs. [3,4] to include damping.

In the present paper, we do this by following the standard Chapman-Enskog procedure, as first generalized for quantum gases by Uehling and Uhlenbeck [7,8]. In Section II, we derive the hydrodynamic equations of a trapped Bose gas above T_{BEC} which include damping due to shear viscosity η and thermal conductivity κ . We also obtain integral equations which give expressions for the transport coefficients η and κ , and in Section III we solve for these. Our approximation corresponds to a lowest-order polynomial approximation of the Chapman-Enskog results such as obtained by Uehling [8]. The transport coefficients have a position dependence when we include the effect of quantum degeneracy of the trapped Bose gas.

In Section IV, we use our results to discuss the hydrodynamic damping of the surface normal modes in a trapped Bose gas discussed in Ref. [3]. In the low density tail of a trapped gas, the hydrodynamic description ceases to be valid. It is crucial to include a cutoff in the linearized hydrodynamic equations to take this into account, as pointed out by Kavoulakis, Petthick and Smith [6,9]. We also explicitly show that with this spatial cutoff, the damping

of normal modes given by our linearized hydrodynamic equations agrees with the expression on which Ref. [6] is based.

In Section V, we extend these results to the Bose-condensed region just below T_{BEC} . Zaremba, Griffin and Nikuni [4] have recently given an explicit closed derivation of two-fluid hydrodynamic equations starting from a microscopic model of a trapped weakly-interacting Bose-condensed gas. We generalize the results of Ref. [4] to include hydrodynamic damping. Finally in Section VI, we give some concluding remarks.

II. CHAPMAN-ENSKOG METHOD

In Section II-IV, we limit ourselves to a non-condensed Bose gas above T_{BEC} , as in Ref. [3]. The atoms are described by the semi-classical kinetic equation for the distribution function $f(\mathbf{r}, \mathbf{p}, t)$ of a Bose gas [10]

$$\left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_r - \nabla U(\mathbf{r}, t) \cdot \nabla_p \right] f(\mathbf{r}, \mathbf{p}, t) = \frac{\partial f}{\partial t} \Big|_{\text{coll}}. \quad (1)$$

Here $U(\mathbf{r}, t) = U_0(\mathbf{r}) + 2gn(\mathbf{r}, t)$ includes an external potential $U_0(\mathbf{r})$ as well as the Hartree-Fock (HF) self-consistent mean field $2gn(\mathbf{r}, t)$, where $n(\mathbf{r}, t)$ is the local density. In usual discussions [7,8], the HF field is omitted but it will play a crucial role when we discuss a Bose-condensed gas below T_{BEC} (see Section V). The quantum collision integral in the right hand side of (1) is given by [10]

$$\begin{aligned} \frac{\partial f}{\partial t} \Big|_{\text{coll}} = & 4\pi g^2 \int \frac{d\mathbf{p}_2}{(2\pi)^3} \int \frac{d\mathbf{p}_3}{(2\pi)^3} \int \frac{d\mathbf{p}_4}{(2\pi)^3} \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta \left(\frac{p^2}{2m} + \frac{p_2^2}{2m} - \frac{p_3^2}{2m} - \frac{p_4^2}{2m} \right) \\ & \times [(1+f)(1+f_2)f_3f_4 - ff_2(1+f_3)(1+f_4)], \end{aligned} \quad (2)$$

where $f \equiv f(\mathbf{r}, \mathbf{p}, t)$, $f_i \equiv f(\mathbf{r}, \mathbf{p}_i, t)$. The interaction strength $g = 4\pi\hbar^2 a/m$ is determined by the s -wave scattering length a .

The conservation laws are obtained by multiplying (1) by 1, \mathbf{p} and p^2 and integrating over \mathbf{p} . In all three cases, the integrals of the collision term in (2) vanish and one finds the general hydrodynamic equations:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \quad (3a)$$

$$mn \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) v_\mu + \frac{\partial P_{\mu\nu}}{\partial x_\nu} + n \frac{\partial U}{\partial x_\mu} = 0, \quad (3b)$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon\mathbf{v}) + \nabla \cdot \mathbf{Q} + D_{\mu\nu} P_{\mu\nu} = 0, \quad (3c)$$

where we have defined the usual quantities:

$$\text{density :} \quad n(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi)^3} f(\mathbf{r}, \mathbf{p}, t), \quad (4)$$

$$\text{velocity :} \quad n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\mathbf{p}}{m} f(\mathbf{r}, \mathbf{p}, t), \quad (5)$$

$$\text{pressure tensor :} \quad P_{\mu\nu}(\mathbf{r}, t) = m \int \frac{d\mathbf{p}}{(2\pi)^3} \left(\frac{p_\mu}{m} - v_\mu \right) \left(\frac{p_\nu}{m} - v_\nu \right) f(\mathbf{r}, \mathbf{p}, t), \quad (6)$$

$$\text{energy density :} \quad \varepsilon(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2m} (\mathbf{p} - m\mathbf{v})^2 f(\mathbf{r}, \mathbf{p}, t), \quad (7)$$

$$\text{heat current :} \quad \mathbf{Q}(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2m} (\mathbf{p} - m\mathbf{v})^2 \left(\frac{\mathbf{p}}{m} - \mathbf{v} \right) f(\mathbf{r}, \mathbf{p}, t), \quad (8)$$

$$\text{rate-of-strain tensor:} \quad D_{\mu\nu} = \frac{1}{2} \left(\frac{\partial v_\mu}{\partial x_\nu} + \frac{\partial v_\nu}{\partial x_\mu} \right). \quad (9)$$

The first approximation for the distribution function is the local equilibrium form

$$f^{(0)}(\mathbf{r}, \mathbf{p}, t) = \{\exp[\beta(\mathbf{p} - m\mathbf{v})^2/2m + U - \mu] - 1\}^{-1}, \quad (10)$$

where the thermodynamic variables β , \mathbf{v} and μ all depend on \mathbf{r} and t , and $U(\mathbf{r}, t)$ has been defined after (1). This expression ensures that the collision integral (2) vanishes. This gives the lowest-order approximation in l/L , where l is the mean free path and L is a characteristic wavelength. If one uses (10) to calculate the quantities from (4) to (8), one finds that $\mathbf{Q} = 0$ and

$$n(\mathbf{r}, t) = \frac{1}{\Lambda^3} g_{3/2}(z(\mathbf{r}, t)), \quad (11)$$

$$P_{\mu\nu}(\mathbf{r}, t) = \delta_{\mu\nu} P(\mathbf{r}, t), \quad P(\mathbf{r}, t) = \frac{k_B T}{\Lambda^3} g_{5/2}(z(\mathbf{r}, t)) = \frac{2}{3} \varepsilon(\mathbf{r}, t). \quad (12)$$

Here $z(\mathbf{r}, t) \equiv e^{\beta(\mathbf{r}, t)[\mu(\mathbf{r}, t) - U(\mathbf{r}, t)]}$ is the local fugacity, $\Lambda(\mathbf{r}, t) = [2\pi\hbar^2/mk_B T(\mathbf{r}, t)]^{1/2}$ is the local thermal de Broglie wavelength and $g_n(z) = \sum_{l=1}^{\infty} z^l/l^n$ are the well-known Bose-Einstein functions. The equilibrium value of the fugacity is given by $z_0(\mathbf{r}) = e^{\beta_0(\mu - U(\mathbf{r}))}$, with $U(\mathbf{r}) = U_0(\mathbf{r}) + 2gn_0(\mathbf{r})$. Putting $\mathbf{Q} = 0$ and using (12) in Eqs. (3), one obtains

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \quad (13a)$$

$$mn \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = - \nabla P - n \nabla U, \quad (13b)$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{5}{3} \nabla \cdot (\varepsilon \mathbf{v}) = \mathbf{v} \cdot \nabla P. \quad (13c)$$

The linearization of the equations in (13) around equilibrium leads to the equations (4-6) in Ref. [3] if we ignore the HF field in $U(\mathbf{r}, t)$.

Solutions of these equations describe undamped oscillations, some example of which are discussed in Ref. [3] (see also Section IV). In order to obtain damping of the oscillations, we have to consider the deviation of the distribution function from the local equilibrium form (10). One assumes a solution of the quantum Boltzmann equation (1) of the form [7,11]:

$$f(\mathbf{r}, \mathbf{p}, t) = f^{(0)}(\mathbf{r}, \mathbf{p}, t) + f^{(0)}(\mathbf{r}, \mathbf{p}, t)[1 + f^{(0)}(\mathbf{r}, \mathbf{p}, t)]\psi(\mathbf{r}, \mathbf{p}, t), \quad (14)$$

where ψ expresses a small deviation from local equilibrium. To first order in ψ , we can reduce the collision integral in (2) to

$$4\pi g^2 \int \frac{d\mathbf{p}_2}{(2\pi)^3} \int \frac{d\mathbf{p}_3}{(2\pi)^3} \int d\mathbf{p}_4 \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta\left(\frac{p^2}{2m} + \frac{p_2^2}{2m} - \frac{p_3^2}{2m} - \frac{p_4^2}{2m}\right) \times f^{(0)} f_2^{(0)} (1 + f_3^{(0)}) (1 + f_4^{(0)}) (\psi_3 + \psi_4 - \psi_2 - \psi) \equiv \hat{L}[\psi], \quad (15)$$

where $\psi_i \equiv \psi(\mathbf{r}, \mathbf{p}_i, t)$. In the left hand side of (1), we approximate f by $f^{(0)}$. The various derivatives of $\mathbf{v}(\mathbf{r}, t)$, $\mu(\mathbf{r}, t)$, $T(\mathbf{r}, t)$ and $U(\mathbf{r}, t)$ with respect to \mathbf{r} and t can be rewritten using the lowest-order hydrodynamic equations given in (13). The resulting linearized equation for ψ is (for details, see Appendix)

$$\left\{ \frac{\mathbf{u} \cdot \nabla T}{T} \left[\frac{mu^2}{2k_B T} - \frac{5g_{5/2}(z)}{2g_{3/2}(z)} \right] + \frac{m}{k_B T} D_{\mu\nu} \left(u_\mu u_\nu - \frac{1}{3} \delta_{\mu\nu} u^2 \right) \right\} f^{(0)} (1 + f^{(0)}) = \hat{L}[\psi], \quad (16)$$

where the thermal velocity \mathbf{u} is defined by $m\mathbf{u} \equiv \mathbf{p} - m\mathbf{v}$ and the strain tensor $D_{\mu\nu}$ is defined in (9). The linearized collision operator \hat{L} is defined by (15). This equation can be shown to have a unique solution for ψ if we impose the constraints

$$\int d\mathbf{p} f^{(0)} (1 + f^{(0)}) \psi = \int d\mathbf{p} p_\mu f^{(0)} (1 + f^{(0)}) \psi = \int d\mathbf{p} p^2 f^{(0)} (1 + f^{(0)}) \psi = 0, \quad (17)$$

which mean physically that n , \mathbf{v} and ε [see (4),(5) and (7)] are determined only by the first term $f^{(0)}$ in (14). Since the left hand of (16) and also $f^{(0)}$ only depend on the relative thermal velocity \mathbf{u} , the function ψ will also depend only on \mathbf{u} (i.e., not separately on \mathbf{p} and \mathbf{v}).

It is convenient to introduce dimensionless velocity variables

$$\left(\frac{m}{2k_B T} \right)^{1/2} \mathbf{u} \equiv \boldsymbol{\xi}. \quad (18)$$

With these dimensionless velocity variables, (16) becomes

$$\frac{\pi^3}{8a^2mk_B^2T^2} \left\{ \left(\frac{2k_BT}{m} \right)^{1/2} \frac{\nabla T \cdot \boldsymbol{\xi}}{T} \left[\xi^2 - \frac{5g_{5/2}(z)}{2g_{3/2}(z)} \right] + 2D_{\mu\nu} \left(\xi_\mu \xi_\nu - \frac{1}{3} \delta_{\mu\nu} \xi^2 \right) \right\} f^{(0)}(\xi) [1 + f^{(0)}(\xi)] = \hat{L}'[\psi], \quad (19)$$

with $f^{(0)}(\xi) = (z^{-1}e^{\xi^2} - 1)^{-1}$ and

$$\begin{aligned} \hat{L}'[\psi] \equiv & \int d\boldsymbol{\xi}_2 \int d\boldsymbol{\xi}_3 \int d\boldsymbol{\xi}_4 \delta(\boldsymbol{\xi} + \boldsymbol{\xi}_2 - \boldsymbol{\xi}_3 - \boldsymbol{\xi}_4) \delta(\xi^2 + \xi_2^2 - \xi_3^2 - \xi_4^2) \\ & \times f^{(0)} f_2^{(0)} (1 + f_3^{(0)}) (1 + f_4^{(0)}) (\psi_3 + \psi_4 - \psi_2 - \psi). \end{aligned} \quad (20)$$

For a more detailed discussion of the mathematical structure of (19) and (20), we refer to the treatment of the analogous equations for classical gases (see for example, Ref. [12]). The most general solution of the integral equation (19) is of the form [7]

$$\psi = \frac{\pi^3}{8a^2mk_B^2T^2} \left[\left(\frac{2k_BT}{m} \right)^{1/2} \frac{\nabla T \cdot \boldsymbol{\xi}}{T} A(\xi) + 2D_{\mu\nu} \left(\xi_\mu \xi_\nu - \frac{1}{3} \delta_{\mu\nu} \xi^2 \right) B(\xi) \right]. \quad (21)$$

The functions $A(\xi)$ and $B(\xi)$ obey the following integral equations:

$$\hat{L}'[\boldsymbol{\xi} A(\xi)] = \boldsymbol{\xi} \left[\xi^2 - \frac{5g_{5/2}(z)}{2g_{3/2}(z)} \right] f^{(0)} (1 + f^{(0)}), \quad (22a)$$

$$\hat{L}' \left[\left(\xi_\mu \xi_\nu - \frac{1}{3} \delta_{\mu\nu} \xi^2 \right) B(\xi) \right] = \left(\xi_\mu \xi_\nu - \frac{1}{3} \delta_{\mu\nu} \xi^2 \right) f^{(0)} (1 + f^{(0)}). \quad (22b)$$

For (21) to satisfy the constraints given in (17), we must also have

$$\int d\boldsymbol{\xi} \xi^2 A(\xi) f^{(0)} (1 + f^{(0)}) = 0. \quad (23)$$

Using a solution of the form (21) in conjunction with (14), one can calculate the heat current density \mathbf{Q} in (8) and the pressure tensor $P_{\mu\nu}$ in (6). One finds these have the form

$$\mathbf{Q} = -\kappa \nabla T, \quad (24a)$$

$$P_{\mu\nu} = \delta_{\mu\nu} P - 2\eta \left[D_{\mu\nu} - \frac{1}{3} (\text{Tr} D) \delta_{\mu\nu} \right], \quad (24b)$$

where the last term in (24b) involves the non-equilibrium stress tensor. The thermal conductivity κ and the shear viscosity coefficient η are given in terms of the functions $A(\xi)$ and $B(\xi)$:

$$\kappa = -\frac{k_B}{48a^2} \left(\frac{2k_B T}{m} \right)^{1/2} \int d\boldsymbol{\xi} \xi^4 A(\boldsymbol{\xi}) f^{(0)}(1 + f^{(0)}), \quad (25a)$$

$$\eta = -\frac{m}{120a^2} \left(\frac{2k_B T}{m} \right)^{1/2} \int d\boldsymbol{\xi} \xi^4 B(\boldsymbol{\xi}) f^{(0)}(1 + f^{(0)}). \quad (25b)$$

Introducing (24) into the general hydrodynamic equations (3), one obtains the following hydrodynamic equations with the effect of shear viscosity and heat conduction included:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{v}) = 0, \quad (26a)$$

$$mn \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \right) v_\mu + \frac{\partial P}{\partial x_\mu} + n \frac{\partial U}{\partial x_\mu} = \frac{\partial}{\partial x_\nu} \left\{ 2\eta \left[D_{\mu\nu} - \frac{1}{3}(\text{Tr} D) \delta_{\mu\nu} \right] \right\}, \quad (26b)$$

$$\frac{\partial \varepsilon}{\partial t} + \boldsymbol{\nabla} \cdot (\varepsilon \mathbf{v}) + (\boldsymbol{\nabla} \cdot \mathbf{v}) P = \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T) + 2\eta \left[D_{\mu\nu} - \frac{1}{3}(\text{Tr} D) \delta_{\mu\nu} \right]^2. \quad (26c)$$

We recall that n, \mathbf{v} and ε are still given by (4)-(7) with $f = f^{(0)}$, which means that the expressions in (11) and (12) for n, ε and P are valid. The form of these equations can be shown to agree with those originally obtained by Uehling and Uhlenbeck [7]. We note that η and κ are position-dependent, but only through their dependence on the equilibrium value of the fugacity $z = z_0(\mathbf{r})$. One slight generalization we have made over the derivation in Ref. [7] is that we have included the Hartree-Fock mean field.

III. THE TRANSPORT COEFFICIENTS

A. The thermal conductivity

To find the thermal conductivity as given by (25a), we can introduce a simple ansatz for the form of the function $A(\boldsymbol{\xi})$ [8,12]:

$$A(\boldsymbol{\xi}) = A \left[\xi^2 - \frac{5g_{5/2}(z)}{2g_{3/2}(z)} \right]. \quad (27)$$

The constant A is determined by multiplying (22a) by $\boldsymbol{\xi}[\xi^2 - 5g_{5/2}(z)/2g_{3/2}(z)]$ and integrating over $\boldsymbol{\xi}$:

$$\begin{aligned} A = & \int d\boldsymbol{\xi} \xi^2 \left[\xi^2 - \frac{5g_{5/2}(z)}{2g_{3/2}(z)} \right]^2 f^{(0)}(1 + f^{(0)}) \\ & \times \left\{ \int d\boldsymbol{\xi} \left(\xi^2 - \frac{5g_{5/2}(z)}{2g_{3/2}(z)} \right) \boldsymbol{\xi} \cdot \hat{L}' \left[\left(\xi^2 - \frac{5g_{5/2}(z)}{2g_{3/2}(z)} \right) \boldsymbol{\xi} \right] \right\}^{-1} \end{aligned}$$

$$= \frac{15\pi^{3/2}}{4I_A} \left[\frac{7}{2}g_{7/2}(z) - \frac{5g_{5/2}^2(z)}{2g_{3/2}(z)} \right], \quad (28)$$

where the integral I_A is defined by

$$I_A \equiv \int d\boldsymbol{\xi} \left[\xi^2 - \frac{5g_{5/2}(z)}{2g_{3/2}(z)} \right] \boldsymbol{\xi} \cdot \hat{L}' \left[\left(\xi^2 - \frac{5g_{5/2}(z)}{2g_{3/2}(z)} \right) \boldsymbol{\xi} \right] = \int d\boldsymbol{\xi} \xi^2 \boldsymbol{\xi} \cdot \hat{L}'[\xi^2 \boldsymbol{\xi}]. \quad (29)$$

In order to evaluate the integral in (29), it is convenient to introduce the change of variables

$$\begin{aligned} \boldsymbol{\xi} &= \frac{1}{\sqrt{2}}(\boldsymbol{\xi}_0 + \boldsymbol{\xi}'), \quad \boldsymbol{\xi}_2 = \frac{1}{\sqrt{2}}(\boldsymbol{\xi}_0 - \boldsymbol{\xi}'), \\ \boldsymbol{\xi}_3 &= \frac{1}{\sqrt{2}}(\boldsymbol{\xi}_0 + \boldsymbol{\xi}''), \quad \boldsymbol{\xi}_4 = \frac{1}{\sqrt{2}}(\boldsymbol{\xi}_0 - \boldsymbol{\xi}''). \end{aligned} \quad (30)$$

Then we introduce transformations from $\xi'_x \xi'_y \xi'_z$ to $\xi' \theta' \phi'$ and $\xi''_x \xi''_y \xi''_z$ to $\xi'' \theta'' \phi''$, where θ', θ'' and ϕ', ϕ'' are the polar and azimuthal angles with respect to the vector $\boldsymbol{\xi}_0$. One obtains the following expression for I_A ,

$$\begin{aligned} I_A &= -4\sqrt{2}\pi^3 I'_A, \\ I'_A(z) &= \int_0^\infty d\xi_0 \int_0^\infty d\xi' \int_0^1 dy' \int_0^1 dy'' \xi_0^4 \xi'^7 F(\xi_0, \xi', y', y''; z) [y'^2 + y''^2 - 2y' y''], \end{aligned} \quad (31)$$

where $y' = \cos \theta', y'' = \cos \theta''$. Here the function F is defined by

$$F \equiv f^{(0)} f_2^{(0)} (1 + f_3^{(0)}) (1 + f_4^{(0)}) = \frac{z^2 e^{-(\xi_0^2 + \xi'^2)}}{(1 - ze^{-\xi^2})(1 - ze^{-\xi_2^2})(1 - ze^{-\xi_3^2})(1 - ze^{-\xi_4^2})}, \quad (32)$$

with

$$\begin{aligned} \xi^2 &= \frac{1}{2}(\xi_0^2 + 2\xi_0 \xi' y' + \xi'^2), \quad \xi_2^2 = \frac{1}{2}(\xi_0^2 - 2\xi_0 \xi' y' + \xi'^2), \\ \xi_3^2 &= \frac{1}{2}(\xi_0^2 + 2\xi_0 \xi' y'' + \xi'^2), \quad \xi_4^2 = \frac{1}{2}(\xi_0^2 - 2\xi_0 \xi' y'' + \xi'^2). \end{aligned} \quad (33)$$

Inserting the expression in (27) into (25a) and carrying out the integration, we obtain the following expression for the thermal conductivity κ :

$$\kappa = -\frac{75k_B}{64a^2m} \left(\frac{mk_B T}{\pi} \right)^{1/2} \frac{\pi^{1/2}}{16I'_A(z)} \left[\frac{7}{2}g_{7/2}(z) - \frac{5g_{5/2}^2(z)}{2g_{3/2}(z)} \right]^2, \quad (34)$$

where the function $I'_A(z)$ is defined in (31).

B. The shear viscosity

In evaluating the shear viscosity in (25b), the simplest consistent approximation [8,12] is to use $B(\xi) \equiv B$. The constant B can be determined by multiplying (22b) by $(\xi_\mu \xi_\nu - \delta_{\mu\nu} \xi^2/3)$ and integrating over ξ ,

$$B = \left\{ \int d\xi \left(\xi_\mu \xi_\nu - \frac{1}{3} \delta_{\mu\nu} \xi^2 \right)^2 f^{(0)}(1 + f^{(0)}) \right\} \left\{ \int d\xi \left(\xi_\mu \xi_\nu - \frac{1}{3} \delta_{\mu\nu} \xi^2 \right) \hat{L}' \left[\xi_\mu \xi_\nu - \frac{1}{3} \delta_{\mu\nu} \xi^2 \right] \right\}^{-1} \\ = \frac{5\pi^{3/2} g_{5/2}(z)}{2I_B}. \quad (35)$$

The function I_B is defined by

$$I_B = \int d\xi \left(\xi_\mu \xi_\nu - \frac{1}{3} \delta_{\mu\nu} \xi^2 \right) \hat{L}' \left[\xi_\mu \xi_\nu - \frac{1}{3} \delta_{\mu\nu} \xi^2 \right] = \int d\xi (\xi_\mu \xi_\nu) \hat{L}'[\xi_\mu \xi_\nu] \\ \equiv -2\sqrt{2}\pi^3 I'_B, \quad (36)$$

where

$$I'_B(z) = \int_0^\infty d\xi_0 \int_0^\infty d\xi' \int_0^1 dy' \int_0^1 dy'' F(\xi_0, \xi', y', y''; z) \xi_0^2 \xi'^7 (1 + y'^2 + y''^2 - 3y'^2 y''^2). \quad (37)$$

This involves the same function F as defined in (32). Using the ansatz $B(\xi) \equiv B$ in conjunction with (35) in (25b), we can carry out the integral. Our final the expression for the viscosity η is

$$\eta = \frac{5}{16} \frac{1}{a^2} \left(\frac{mk_B T}{\pi} \right)^{1/2} \frac{\pi^{1/2}}{8I'_B(z)} g_{5/2}^2(z). \quad (38)$$

C. High-temperature limit

The formulas in (34) and (38) give κ and η as a function of the fugacity z . We note that the integrals I'_A and I'_B also depend on z . These four-dimensional integrals in (31) and (37) can be evaluated numerically. In Figs. 1 and 2, we plot both κ and η as a function of the fugacity z . These graphs are valid for any trapping potential $U_0(\mathbf{r})$ since the latter only enters into the equilibrium fugacity $z_0 = \exp[\beta_0(\mu - U_0(\mathbf{r}) - 2gn_0(\mathbf{r}))]$.

In the high-temperature limit, where the fugacity $z = e^{\beta(\mu - U(\mathbf{r}))}$ is small, the local distribution function $f^{(0)}$ can be expanded in terms of z . To third order in z , the function F in (32) reduces to

$$F = z^2 e^{-(\xi_0^2 + \xi'^2)} + 2z^3 e^{-\frac{3}{2}(\xi_0^2 + \xi'^2)} [\cosh(\xi_0 \xi' y') + \cosh(\xi_0 \xi' y'')] + O(z^4). \quad (39)$$

The integrals in (31) and (37) can be evaluated analytically,

$$I'_B(z) = 2I'_A(z) = \pi^{1/2} z^2 \left(1 + z \frac{9}{16} \sqrt{\frac{3}{2}} \right). \quad (40)$$

We thus obtain the following explicit expressions for the transport coefficients to first order in the fugacity z :

$$\begin{aligned} \kappa &= \frac{1}{8} \left(\frac{75}{64} \right) \frac{k_B}{a^2} \left(\frac{k_B T}{\pi m} \right)^{1/2} \left[1 + z \left(\frac{7\sqrt{2}}{16} - \frac{9}{16} \sqrt{\frac{3}{2}} \right) \right] \\ &\approx \frac{1}{8} \left(\frac{75}{64} \right) \frac{k_B}{a^2} \left(\frac{k_B T}{\pi m} \right)^{1/2} (1 - 0.07 n_0 \Lambda^3), \end{aligned} \quad (41a)$$

$$\begin{aligned} \eta &= \frac{1}{8} \left(\frac{5}{16} \right) \frac{m}{a^2} \left(\frac{k_B T}{\pi m} \right)^{1/2} \left[1 + z \left(\frac{1}{2\sqrt{2}} - \frac{9}{16} \sqrt{\frac{3}{2}} \right) \right] \\ &\approx \frac{1}{8} \left(\frac{5}{16} \right) \frac{m}{a^2} \left(\frac{k_B T}{\pi m} \right)^{1/2} (1 - 0.335 n_0 \Lambda^3). \end{aligned} \quad (41b)$$

The local equilibrium density $n_0(\mathbf{r})$ in the high-temperature limit is the classical result $n_0 = \Lambda^{-3} e^{\beta(\mu - U(\mathbf{r}))}$. The terms first order in z in (41a) and (41b) give the first order corrections to the classical results due to Bose statistics. If we ignore the HF mean field $2gn_0(\mathbf{r})$, these transport coefficients reduce to the expressions first obtained for a uniform Bose gas by Uehling [8]. The Bose quantum corrections to the classical results in both η and κ depend on the local fugacity z and, through this, on position in a trapped gas.

The density-independent terms in Eqs. (41a) and (41b) are 8 times smaller than the well-known Chapman-Enskog expressions for classical hard spheres. This is due to the difference (see also Ref. [8]) in the quantum binary scattering cross-section for Bosons when correctly calculated using symmetrized wavefunctions ($\sigma = 8\pi a^2$, instead of πa^2).

IV. HYDRODYNAMIC DAMPING OF NORMAL MODES

The linearized version of the hydrodynamic equations in (26) are

$$\frac{\partial \delta n}{\partial t} + \nabla \cdot (n_0 \delta \mathbf{v}) = 0, \quad (42a)$$

$$mn_0 \frac{\partial \delta v_\mu}{\partial t} = -\frac{\partial \delta P}{\partial x_\mu} - \delta n \frac{\partial U}{\partial x_\mu} - 2gn_0 \frac{\partial \delta n}{\partial x_\mu} + \frac{\partial}{\partial x_\nu} \left\{ 2\eta \left[D_{\mu\nu} - \frac{1}{3} (\text{Tr} D) \delta_{\mu\nu} \right] \right\}, \quad (42b)$$

$$\frac{\partial \delta P}{\partial t} = -\frac{5}{3} \nabla \cdot (P_0 \delta \mathbf{v}) + \frac{2}{3} \delta \mathbf{v} \cdot \nabla P_0 + \frac{2}{3} \nabla \cdot (\kappa \nabla T), \quad (42c)$$

where repeated Greek subscripts are summed over.

Taking the time derivative of (42b) and using (42a) and (42c) gives an equation for the velocity fluctuations which is only coupled to the temperature fluctuations:

$$\begin{aligned}
m \frac{\partial^2 \delta v_\mu}{\partial t^2} = & \frac{5P_0}{3n_0} \frac{\partial}{\partial x_\mu} (\nabla \cdot \delta \mathbf{v}) - \frac{\partial}{\partial x_\mu} (\delta \mathbf{v} \cdot \nabla U_0) - \frac{2}{3} (\nabla \cdot \delta \mathbf{v}) \frac{\partial U_0}{\partial x_\mu} \\
& + 2g \frac{\partial}{\partial x_\mu} [n_0 (\nabla \cdot \delta \mathbf{v})] - \frac{4}{3} g (\nabla \cdot \delta \mathbf{v}) \frac{\partial}{\partial x_\mu} n_0 \\
& + \frac{1}{n_0} \frac{\partial}{\partial x_\nu} \left\{ 2\eta \left[\frac{\partial}{\partial t} D_{\mu\nu} - \frac{1}{3} \left(\text{Tr} \frac{\partial D}{\partial t} \right) \delta_{\mu\nu} \right] \right\} - \frac{2}{3n_0} \frac{\partial}{\partial x_\mu} \nabla \cdot (\kappa \nabla \delta T). \quad (43)
\end{aligned}$$

Here we have used the relation $\nabla P_0 = -n_0 \nabla U$ valid in equilibrium. Since we assume the dissipative terms to be small, we can use the lowest-order hydrodynamic equation for δT in (43) (see (A7) in Appendix),

$$\frac{\partial \delta T}{\partial t} = -\frac{2}{3} T_0 \nabla \cdot \delta \mathbf{v}. \quad (44)$$

Equations (43) and (44) are thus a closed set of equations for the fluctuations. Equation (43) generalizes that given by (13) in Ref. [3] to include damping due to viscosity and thermal conductivity as well as the HF mean field $2gn$.

At this point, we introduce a crucial aspect of a trapped gas which leads to an important modification of the preceding hydrodynamic equations. Kavoulakis et al. [6] have pointed out the breakdown of the local equilibrium (hydrodynamic) description in the low density outer region of a trapped gas. The corrections involving thermal conduction and viscous processes are expected to vanish (over a mean free path) in the dilute region where collisions become ineffective in producing local equilibrium. This effect can be simulated in the hydrodynamic equations by multiplying the hydrodynamic expression for the thermal conductivity in (24a) and the viscous stress tensor in (24b) by a step function that vanishes outside the cloud [9]:

$$\mathbf{Q} = -\kappa \nabla T \Theta(r_0 - r), \quad (45a)$$

$$P_{\mu\nu} = \delta_{\mu\nu} P - 2\eta \left[D_{\mu\nu} - \frac{1}{3} (\text{Tr} D) \delta_{\mu\nu} \right] \Theta(r_0 - r), \quad (45b)$$

For simplicity of notation, we only consider an isotropic trap, in which case the hydrodynamic description breaks down on a sphere of radius r_0 . A method of calculating r_0 is given

in Ref. [6] for a classical trapped gas. The equation of motion in (43) for the velocity fluctuations is modified to

$$\begin{aligned}
m \frac{\partial^2 \delta v_\mu}{\partial t^2} = & \frac{5P_0}{3n_0} \frac{\partial}{\partial x_\mu} (\nabla \cdot \delta \mathbf{v}) - \frac{\partial}{\partial x_\mu} (\delta \mathbf{v} \cdot \nabla U_0) - \frac{2}{3} (\nabla \cdot \delta \mathbf{v}) \frac{\partial U_0}{\partial x_\mu} \\
& + 2g \frac{\partial}{\partial x_\mu} [n_0 (\nabla \cdot \delta \mathbf{v})] - \frac{4}{3} g (\nabla \cdot \delta \mathbf{v}) \frac{\partial n_0}{\partial x_\mu} \\
& + \frac{1}{n_0} \frac{\partial}{\partial x_\nu} \left\{ 2\eta \left[\frac{\partial}{\partial t} D_{\mu\nu} - \frac{1}{3} \left(\text{Tr} \frac{\partial D}{\partial t} \right) \delta_{\mu\nu} \right] \Theta(r_0 - r) \right\} \\
& - \frac{2}{3n_0} \frac{\partial}{\partial x_\mu} \nabla \cdot [\kappa \nabla \delta T \Theta(r_0 - r)].
\end{aligned} \tag{46}$$

We now look for normal mode solutions of (46) with the convention $\delta \mathbf{v}(\mathbf{r}, t) = \delta \mathbf{v}_\omega(\mathbf{r}) e^{-i\omega t}$, $\delta T(\mathbf{r}, t) = \delta T_\omega(\mathbf{r}) e^{-i\omega t}$ and $D(\mathbf{r}, t) = D_\omega(\mathbf{r}) e^{-i\omega t}$. Combining (44) and (46), we obtain a closed equation for $\delta \mathbf{v}_\omega$:

$$\begin{aligned}
& -m n_0 (\omega^2 \delta v_{\omega\mu} - \hat{W}_\mu [\delta \mathbf{v}_\omega]) \\
= & -i\omega \frac{\partial}{\partial x_\nu} \left\{ 2\eta \left[D_{\omega\mu\nu} - \frac{1}{3} (\text{Tr} D_\omega) \delta_{\mu\nu} \right] \Theta(r_0 - r) \right\} \\
& + i \frac{4T_0}{9n_0\omega} \frac{\partial}{\partial x_\mu} \{ \nabla \cdot [\kappa \nabla (\nabla \cdot \delta \mathbf{v}_\omega) \Theta(r_0 - r)] \},
\end{aligned} \tag{47}$$

where the operator $\hat{W}_\mu [\delta \mathbf{v}_\omega]$ is defined by

$$\begin{aligned}
-m \hat{W}_\mu [\delta \mathbf{v}_\omega] \equiv & \frac{5P_0}{3n_0} \frac{\partial}{\partial x_\mu} (\nabla \cdot \delta \mathbf{v}_\omega) - \frac{\partial}{\partial x_\mu} (\delta \mathbf{v}_\omega \cdot \nabla U_0) - \frac{2}{3} (\nabla \cdot \delta \mathbf{v}_\omega) \frac{\partial U_0}{\partial x_\mu} \\
& + 2g \frac{\partial}{\partial x_\mu} [n_0 (\nabla \cdot \delta \mathbf{v}_\omega)] - \frac{4}{3} g (\nabla \cdot \delta \mathbf{v}_\omega) \frac{\partial n_0}{\partial x_\mu}.
\end{aligned} \tag{48}$$

The undamped solution $\delta \tilde{\mathbf{v}}_\omega$ of (47) when $\kappa, \eta = 0$ is given by $\hat{W}_\mu [\delta \tilde{\mathbf{v}}_\omega] = \tilde{\omega}^2 \delta \tilde{v}_{\omega\mu}$ (a tilde denotes undamped solution). For later use, we note that the operator \hat{W}_μ has the following property

$$\int n_0 \delta v_{\omega'\mu}^* \hat{W}_\mu [\delta \mathbf{v}_\omega] d\mathbf{r} = \int n_0 \delta v_{\omega\mu} \hat{W}_\mu [\delta \mathbf{v}_{\omega'}^*] d\mathbf{r}. \tag{49}$$

Therefore the undamped solutions $\delta \tilde{\mathbf{v}}_\omega$ satisfy the following orthogonality relation

$$\int n_0 \delta \tilde{v}_{\omega'\mu}^* \hat{W}_\mu [\delta \tilde{\mathbf{v}}_\omega] d\mathbf{r} = 0, \quad \text{if } \tilde{\omega}' \neq \tilde{\omega}. \tag{50}$$

We consider an isotropic trap $U_0(\mathbf{r}) = \frac{1}{2} m \omega_0^2 r^2$. The undamped solution for a monopole or breathing mode is described by $\delta \tilde{\mathbf{v}}_\omega(\mathbf{r}) \sim \mathbf{r}$, the undamped frequency is found to be

$\tilde{\omega} = 2\omega_0$ [3]. Using this solution in (47), we find $D_{\omega\mu\nu} \sim \delta_{\mu\nu}$ and hence there is no shear viscous contribution to the damping of this mode. In addition, since $\nabla \cdot \delta\tilde{\mathbf{v}}_\omega = \text{const}$, the last term in (47) involving the thermal conductivity coefficient κ does not contribute. We conclude the *monopole mode has no damping* for an isotropic harmonic trap. As noted in Ref. [3], this surprising result was first derived for a classical gas by Boltzmann in 1876.

The undamped solution for divergence-free surface modes ($\nabla \cdot \delta\tilde{\mathbf{v}}_\omega = 0$) is given by [3] $\delta\tilde{\mathbf{v}}_\omega(\mathbf{r}) = \nabla\chi_\omega(\mathbf{r})$, where

$$\chi_\omega(\mathbf{r}) \propto r^l Y_{lm}(\theta, \phi), \quad (51)$$

with the dispersion relation $\tilde{\omega} = \sqrt{l}\omega_0$, $l = 1, 2, \dots$. Using the solution (51) in the right hand side of (47), we find (there is no contribution from the thermal conductivity because $\nabla \cdot \delta\tilde{\mathbf{v}}_\omega = 0$).

$$\begin{aligned} \text{RHS of (47)} &= -2i\omega \frac{\partial}{\partial x_\nu} \left[\eta \frac{\partial^2 \chi_\omega}{\partial x_\mu \partial x_\nu} \Theta(r_0 - r) \right] \\ &= 2i\omega \left[\frac{z}{k_B T} \frac{\partial \eta}{\partial z} m\omega_0^2 \Theta(r_0 - r) + \frac{\eta}{r} \delta(r - r_0) \right] x_\nu \frac{\partial^2 \chi_\omega}{\partial x_\mu \partial x_\nu} \\ &= 2i\omega(l-1) \left[\frac{z}{k_B T} \frac{\partial \eta}{\partial z} m\omega_0^2 \Theta(r_0 - r) + \frac{\eta}{r_0} \delta(r - r_0) \right] \delta v_{\omega\mu}. \end{aligned} \quad (52)$$

For $l = 1$, we see that the right hand side of (47) vanishes. This means that the center-of-mass mode with frequency ω_0 has no hydrodynamic damping. For $l > 1$, the undamped solution is no longer the solution of the equation of motion (47). In this case, Eq. (47) must be solved for $r < r_0$ with a boundary condition at $r = r_0$ to take into account the discontinuity at r_0 given in (52).

To complete this section, we derive a general expression for hydrodynamic damping from the linearized hydrodynamic equations for the velocity fluctuations given by (47). Multiplying (47) by $\delta v_{\omega\mu}^*$ and integrating over \mathbf{r} , we obtain

$$\begin{aligned} &-m\omega^2 \int n_0 |\delta v_\omega|^2 d\mathbf{r} + m \int n_0 \delta v_{\omega\mu}^* \hat{W}_\mu [\delta \mathbf{v}_\omega] d\mathbf{r} \\ &= -i\omega \int \delta v_{\omega\mu}^* \frac{\partial}{\partial x_\nu} \left\{ 2\eta \left[D_{\omega\mu\nu} - \frac{1}{3} (\text{Tr} D_\omega) \delta_{\mu\nu} \right] \Theta(r_0 - r) \right\} d\mathbf{r} \\ &\quad + \frac{4T_0}{9n_0\omega} \int \frac{\partial}{\partial x_\mu} \{ \nabla \cdot [\kappa \nabla (\nabla \cdot \delta \mathbf{v}_\omega) \Theta(r_0 - r)] \} d\mathbf{r} \\ &= i\omega \int \left[2\eta \left| D_{\omega\mu\nu} - \frac{1}{3} (\text{Tr} D_\omega) \delta_{\mu\nu} \right|^2 + \frac{4T_0}{9n_0\omega^2} \kappa |\nabla (\nabla \cdot \delta \mathbf{v}_\omega)|^2 \right] \Theta(r_0 - r) d\mathbf{r}. \end{aligned} \quad (53)$$

Since the undamped solutions $\delta\tilde{\mathbf{v}}_\omega$ satisfy the orthogonality relation (50), one sees that these undamped solutions can be used to find the eigenvalue ω^2 of (53) to first order in η and κ . Inserting the undamped solutions into (53), we find it reduces

$$\begin{aligned} & -m(\omega^2 - \tilde{\omega}^2) \int n_0 |\delta\tilde{v}_\omega|^2 d\mathbf{r} \\ & = i\omega \int \left[2\eta \left| \tilde{D}_{\omega\mu\nu} - \frac{1}{3}(\text{Tr}\tilde{D}_\omega)\delta_{\mu\nu} \right|^2 + \frac{4T_0}{9n_0\omega^2}\kappa |\nabla(\nabla \cdot \delta\tilde{\mathbf{v}}_\omega)|^2 \right] \Theta(r_0 - r) d\mathbf{r}. \end{aligned} \quad (54)$$

Assuming the complex solution $\omega = \tilde{\omega} - i\Gamma$, one finds the damping rate Γ is given by the expression (valid to first order in η and κ)

$$\begin{aligned} \Gamma & = \int \left[2\eta \left| \tilde{D}_{\omega\mu\nu} - \frac{1}{3}(\text{Tr}\tilde{D}_\omega)\delta_{\mu\nu} \right|^2 + \frac{4T_0}{9n_0\tilde{\omega}^2}\kappa |\nabla(\nabla \cdot \delta\tilde{\mathbf{v}}_\omega)|^2 \right] \Theta(r_0 - r) d\mathbf{r} \\ & \quad \times \left[2m \int n_0 |\delta\tilde{v}_\omega|^2 d\mathbf{r} \right]^{-1}. \end{aligned} \quad (55)$$

The damping rate given by (55) is precisely [9] the one obtained in Ref. [6], where the damping rate is calculated from the rate of increase of the total entropy [14]. The transport coefficients κ and η in (55) are given in Figs. 1 and 2 for a trapped Bose gas and in general are dependent on position through the fugacity.

V. GENERALIZATION TO A SUPERFLUID GAS

In this section, we extend the hydrodynamic equations derived in Section II to the case of a superfluid trapped gas by generalizing the results of Ref. [4] to include hydrodynamic damping. The two-fluid equations in Ref. [4] consist of equations of motion for the condensate and a set of hydrodynamic equations for the non-condensate. The latter description is only valid for in the semiclassical region $k_B T \gg \hbar\omega_0, gn_0$. The time-dependent Hartree-Fock-Popov equation of motion for the condensate wavefunction can be rewritten in terms of a pair of hydrodynamic equations for the condensate density n_c and the superfluid velocity \mathbf{v}_s ,

$$\frac{\partial n_c}{\partial t} = -\nabla \cdot (n_c \mathbf{v}_s), \quad (56a)$$

$$m \left(\frac{\partial \mathbf{v}_s}{\partial t} + \frac{1}{2} \nabla \mathbf{v}_s^2 \right) = -\nabla \phi. \quad (56b)$$

The potential ϕ is defined by [4]

$$\phi(\mathbf{r}, t) \equiv -\frac{\hbar^2 \nabla^2 [n_c(\mathbf{r}, t)]^{1/2}}{2m[n_c(\mathbf{r}, t)]^{1/2}} + U_0(\mathbf{r}) + 2g\tilde{n}(\mathbf{r}, t) + gn_c(\mathbf{r}, t), \quad (57)$$

where $\tilde{n}(\mathbf{r}, t)$ is the non-condensate density describing the excited atoms. The first term in (57) is sometimes called the quantum pressure term and vanishes in uniform systems. It is associated with the spatially varying amplitude of the Bose order parameter.

The equations of motion for the non-condensate can be derived from the kinetic equation (1) for the distribution function $f(\mathbf{r}, \mathbf{p}, t)$ of the excited atoms. The HF mean field in $U(\mathbf{r}, t)$ is now $2gn(\mathbf{r}, t) = 2g[\tilde{n}(\mathbf{r}, t) + n_c(\mathbf{r}, t)]$, where $\tilde{n}(n_c)$ represents the local density of the non-condensate (condensate) atoms. Apart from this change, the expression for the collision integral in (2) is still applicable at finite temperatures below T_{BEC} . Thus the hydrodynamic equations for the non-condensate can be derived using precisely the same procedure developed in Section II for $T > T_{\text{BEC}}$. We obtain equations analogous to (26) in the following form:

$$\frac{\partial \tilde{n}}{\partial t} + \nabla \cdot (\tilde{n} \mathbf{v}_n) = 0, \quad (58a)$$

$$m\tilde{n} \left(\frac{\partial}{\partial t} + \mathbf{v}_n \cdot \nabla \right) v_{n\mu} + \frac{\partial \tilde{P}}{\partial x_\mu} + \tilde{n} \frac{\partial U}{\partial x_\mu} = \frac{\partial}{\partial x_\nu} \left\{ 2\eta \left[D_{\mu\nu} - \frac{1}{3}(\text{Tr} D) \delta_{\mu\nu} \right] \right\}, \quad (58b)$$

$$\frac{\partial \tilde{\varepsilon}}{\partial t} + \nabla \cdot (\tilde{\varepsilon} \mathbf{v}_n) + (\nabla \cdot \mathbf{v}_n) \tilde{P} = \nabla \cdot (\kappa \nabla T) + 2\eta \left[D_{\mu\nu} - \frac{1}{3}(\text{Tr} D) \delta_{\mu\nu} \right]^2. \quad (58c)$$

Here \tilde{n} , \tilde{P} and $\tilde{\varepsilon}$ are given by the same expressions as n , P and ε in (11) and (12). $D_{\mu\nu}$ is defined as in (9), where now the velocity \mathbf{v} is the normal fluid velocity \mathbf{v}_n . The expressions for the transport coefficients given in (34) and (38) are directly applicable below T_{BEC} for any trapping potential, apart from change in the form of the HF field in the local fugacity z . As in the case above T_{BEC} considered in Section IV, one must introduce a cutoff in the linearized hydrodynamic equations to take into account that the terms proportional to the transport coefficients vanish when the density becomes too low to sustain local equilibrium.

We might note that the results for κ and η by Kirkpatrick and Dorfman [11,15] for a uniform gas below T_{BEC} have an additional factor $(1 + bn_c \Lambda^3)^{-1}$, where b is of order 1. This factor arises from collisions between condensate and excited atoms, which are important at very low temperatures ($T \ll T_{\text{BEC}}$). Our present analysis ignores this contribution. In this regard, we note that (1) and (2) are based on validity of a simple semi-classical Hartree-Fock description of excited atoms (ignoring the off-diagonal or anomalous self-energy terms). This

has been shown [16] to give an excellent approximation for the thermodynamic properties of a trapped Bose gas down to much lower temperatures ($T \ll T_{\text{BEC}}$) than in a uniform gas.

If we neglect the first term in (57), the equations in (56) and (58) can be used to derive [4] the two-fluid equations for a trapped Bose-condensed gas in the Landau form [17], except that our equations do not include any effect from the bulk viscosities. In a uniform superfluid Bose gas, Kirkpatrick and Dorfman [11] found that the bulk viscosities vanish in the finite temperature region ($k_{\text{B}}T \gg gn_0$) we have been considering. Our present results for a trapped Bose gas are consistent with this and moreover justify the omission of any dissipative terms in the condensate equation of motion (56b).

VI. CONCLUDING REMARKS

Summarizing our main results, starting from the microscopic equations of motion, we have derived a closed equation (43) for the velocity fluctuations of a trapped Bose gas, which includes damping due to hydrodynamic processes. We have also obtained explicit expressions for the shear viscosity η and the thermal conductivity κ for a trapped Bose gas, which depend on position through the local fugacity z_0 . We have given a detailed derivation for a trapped gas above T_{BEC} , and more briefly discussed the generalization of these results to a Bose-condensed gas below T_{BEC} in Section V.

For illustration, in Section IV, we used the linearized hydrodynamic equations to discuss some of the normal modes of oscillation for a normal Bose gas above T_{BEC} . We show that the monopole mode of an isotropic trap was undamped. We also note that our linearized equations in (42b) and (42c) do not take into account that in the low density region in trapped Bose gases, the hydrodynamic description breaks down, as emphasized in Refs. [6,9]. We show that if a spatial cutoff is used in our linearized hydrodynamic equations, the damping of the normal mode solutions is given by the same expression as one finds from calculating the rate of entropy production [6,9]. In this formalism, one has reduced the calculation of the damping to that of determining the boundary where hydrodynamics breaks down. This has been worked out in Ref. [6] for a classical trapped gas, in which the transport coefficients are independent of density. An interesting problem for the future is to use the semi-classical kinetic equation given by (1) and (2) to give a more microscopic analysis of

the low density region where expanding around the local equilibrium solution [see (14)] is no longer valid.

We note that the general form of the hydrodynamic equations given in Eq. (26) can be derived more generally using conservation laws for conserved quantities in conjunction with local thermal equilibrium. This approach is developed in many textbooks [18] and can be formally extended to deal with inhomogeneous superfluids in an external potential (see, for example, Ref. [19]). This method is, of course, phenomenological in that it introduces various linear response coefficients [relating densities (thermodynamic derivatives) and currents (transport coefficients) to fields] whose evaluation requires some specific microscopic model. While very useful, this approach must be ultimately justified by a fully microscopic treatment given by the underlying kinetic equations, such as used in this paper and in Refs. [3,4,7,11]. In particular, as noted above, such a microscopic approach is needed to deal with the breakdown in the local equilibrium solution in the low density tail of trapped gases. Moreover we note that our microscopic derivation in Section V (see also Ref. [4]) gives a more complete description, as can be seen from the fact that we obtain separate conservation laws for the condensate and non-condensate densities. The standard phenomenological two-fluid equations do not determine the condensate and non-condensate density fluctuations separately, but only the total density fluctuation.

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APPENDIX A:

We briefly sketch the derivation of the left hand side of the kinetic equation in (16). Using (10) in the left hand side of (1), one has

$$\begin{aligned}
& \left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_r - \nabla_r U(\mathbf{r}, t) \cdot \nabla_p \right] f^{(0)}(\mathbf{r}, \mathbf{p}, t) \\
&= \left[\frac{1}{z} \left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_r \right) z + \frac{mv^2}{2k_B T^2} \left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_r \right) T \right. \\
&\quad \left. + \frac{m\mathbf{u}}{k_B T} \cdot \left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_r \right) \mathbf{v} + \frac{\nabla_r U(\mathbf{r}, t)}{k_B T} \cdot \mathbf{u} \right] f^{(0)}(1 + f^{(0)}). \tag{A1}
\end{aligned}$$

Using the lowest-order hydrodynamic equations in (13) and the expressions for the density n in (11) and the pressure P in (12), one finds

$$\frac{\partial n}{\partial t} = \frac{3n}{2T} \frac{\partial T}{\partial t} + \frac{\gamma k_B T}{z} \frac{\partial z}{\partial t} \tag{A2}$$

$$= -\frac{3n}{2T} \mathbf{v} \cdot \nabla T - \frac{\gamma k_B T}{z} \mathbf{v} \cdot \nabla z - n \nabla \cdot \mathbf{v}, \tag{A3}$$

where $\gamma \equiv \frac{1}{k_B T} \frac{1}{\Lambda^3} g_{1/2}(z)$, and

$$\frac{\partial P}{\partial t} = \frac{5P}{2T} \frac{\partial T}{\partial t} + \frac{nk_B T}{z} \frac{\partial z}{\partial t} \tag{A4}$$

$$= -\frac{5P}{2T} \mathbf{v} \cdot \nabla T - \frac{nk_B T}{z} \mathbf{v} \cdot \nabla z - \frac{5}{3} P (\nabla \cdot \mathbf{v}). \tag{A5}$$

One may combine these equations to obtain

$$\frac{\partial z}{\partial t} = -\mathbf{v} \cdot \nabla z = -\frac{z}{nk_B T} \mathbf{v} \cdot \left(\nabla P - \frac{5P}{2T} \nabla T \right), \tag{A6}$$

$$\frac{\partial T}{\partial t} = -\frac{2}{3} T \nabla \cdot \mathbf{v} - (\mathbf{v} \cdot \nabla) T. \tag{A7}$$

The analogous equation for $\partial \mathbf{v} / \partial t$ is given directly by (13b). Using these results in (A1), one finds that it reduces to

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_r - \nabla_r U \cdot \nabla_p \right) f^{(0)} \\
&= \left\{ \frac{1}{T} \mathbf{u} \cdot \nabla T \left(\frac{mu^2}{2k_B T} - \frac{5P}{2nk_B T} \right) + \frac{m}{k_B T} \left[\mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \mathbf{v} - \frac{u^2}{3} \nabla \cdot \mathbf{v} \right] \right\} f^{(0)}(1 + f^{(0)}), \tag{A8}
\end{aligned}$$

where we recall $\mathbf{u} \equiv \mathbf{p}/m - \mathbf{v}$. This can be rewritten in the form shown on the left hand side of (16).

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FIGURE CAPTIONS

Fig.1: The thermal conductivity κ as a function of the fugacity z . The values are normalized to the classical gas results at $z = 0$. (see (41a))

Fig.2: The viscosity coefficient η as a function of the fugacity z . The values are normalized to the classical gas results at $z = 0$. (see (41b))

Figure 1

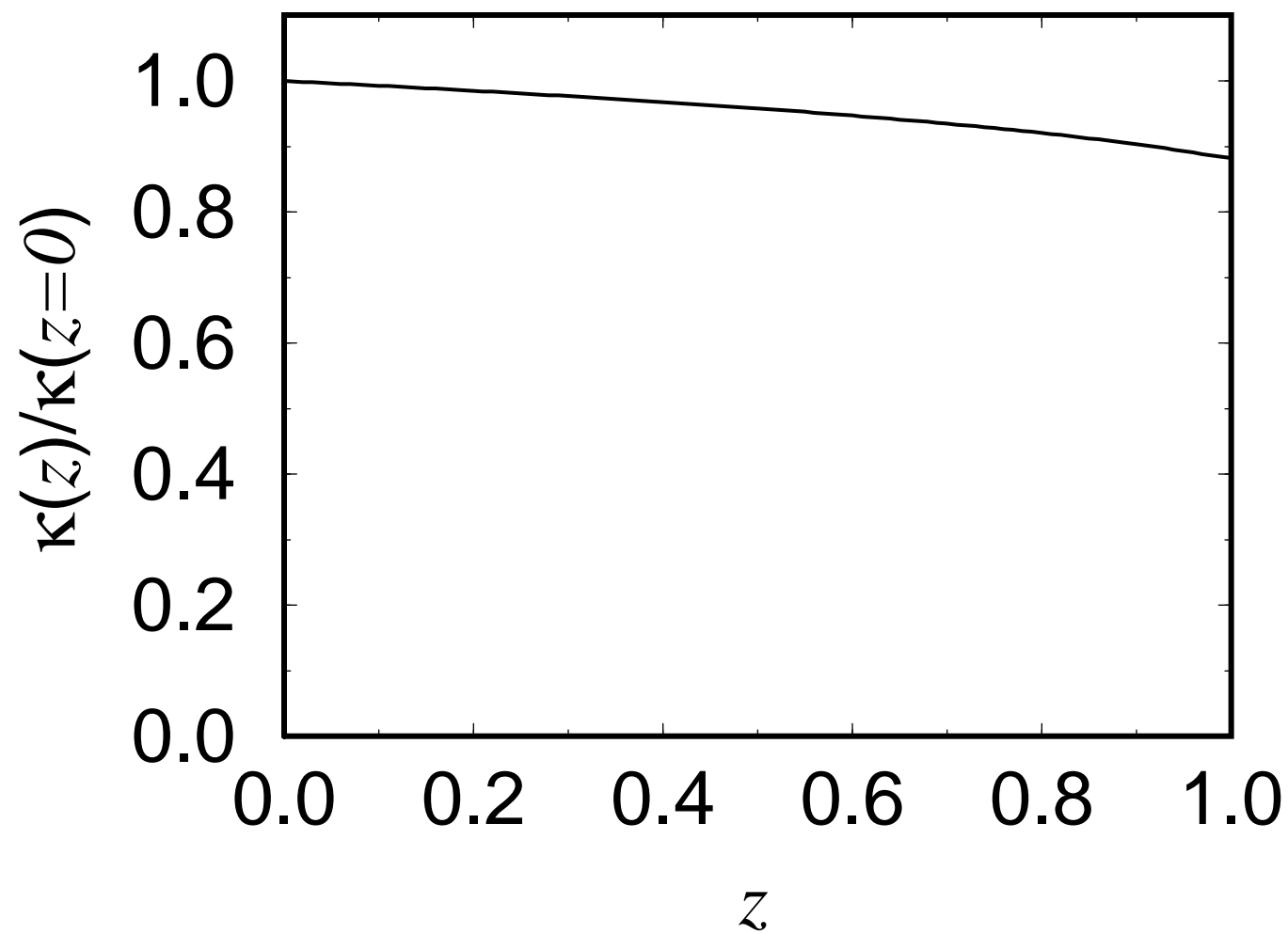


Figure 2

